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On the variation of G and scale invariant gravitation – a reply

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Summary. The scale invariant gravitational Lagrangian $\mathscr{L}_{\mathrm{g}}$ presented by the author several years ago, must be supplemented by a matter Lagrangian $\mathscr{L}_{\mathrm{m}}$.

It is shown that the form of $\mathscr{L}_{\mathrm{m}}$ proposed by Kembhavi & Pollock (their equation 3) far from being 'inescapable' or 'the only possible form' as the authors claim, is instead a very particular case that *contains already imprinted* all the results KP believe they have discovered.

The conclusions of KP are therefore erroneous.

In 1977 Canuto *et al.* proposed a formalism to deal with the possible difference between gravitational and atomic clocks (or times). The formalism was incomplete since the authors only dealt with the gravitational Lagrangian $\mathscr{L}_{\mathrm{g}}$, leaving the specific form of the matter Lagrangian $\mathscr{L}_{\mathrm{m}}$ for a later study.

Kembhavi & Pollock (1981, referred to as KP) have now proposed a matter Lagrangian $\mathscr{L}_{\mathrm{m}}$ which they define alternatively as 'the only possible one' or the 'inescapable form of $\mathscr{L}_{\mathrm{m}}$'.

Adding this $\mathscr{L}_{\mathrm{m}}$ to $\mathscr{L}_{\mathrm{g}}$, they then claim that the theory does not allow for a variable G.

It is the purpose of this note to present a mathematical proof that the proposed matter Lagrangian $\mathscr{L}_{\mathrm{m}}$, far from being 'the only one' or the 'inescapable form' is actually a very special case which *already contains imprinted all the conclusions KP claim to have discovered.* Their results are therefore erroneous.

The critical quantity is their equation (3)

$$\mathscr{L}_{\mathrm{m}} = \text{constant} \int \beta \, ds \tag{1}$$

which KP characterize as 'the only possible matter Lagrangian'. This is an incorrect claim since the most general scale invariant $\mathscr{L}_{\mathrm{m}}$ is instead

$$\mathscr{L}_{\mathrm{m}} = \int m\beta^{2-g} \, ds \tag{2}$$

which is written neither in atomic nor in gravitational units, but in *general units.* Only in this case does it make any sense to talk about *scale invariance.* The quantity m in (2) is not in general constant, neither is it to be taken to be either the mass of a single particle or the mass of a macroscopic object. It is a general symbol for a mass. Going from Einstein units to *general units* we write,

$$ds_{\mathrm{E}} = \beta(x)ds, \qquad A_{\mathrm{E}} = \beta^{\pi(A)}A, \tag{3}$$

where by A we mean any physical quantity and by $\pi(A)$ we indicate its power under scale transformations. Since GM has the units of length, we have ($M = mN$, where N being a pure number, has zero power)

$$\pi(m) = 1 - \pi(G) \equiv 1 - g, \qquad m_{\mathrm{E}} = \beta^{1-g}m. \tag{4}$$

Since by definition the power of β is -1, we now have from (2)

$$\pi(\mathscr{L}_{\mathrm{m}}) = \pi(m) + \pi(\beta^{2-g}) + \pi(ds) = 0 \tag{5}$$

i.e. the power of $\mathscr{L}_{\mathrm{m}}$ is zero, which is the only way it makes sense to talk about scale invariance: a quantity is truly scale invariant if its power under the first equation of (3) is zero. Now, KP's choice

$$\mathscr{L}_{\mathrm{m}} = \text{constant} \int \beta\, ds \tag{6}$$

is a particular case of (2) if *we impose*

$$m\beta^{1-g} = \text{constant}. \tag{7}$$

This is the condition KP do not realize they have actually included and which is fully responsible for their final result. Because of the general relation (4), the assumption (7) implies

$$m_{\mathrm{E}} = \text{constant}. \tag{8}$$

So far we have been working in *general units*, and derived the limitation implied by the choice (7). Let us now go to atomic units, and let us specialize the mass m to be the mass of a single particle. By definition of atomic units, we shall then have

$$m = m_{\mathrm{a}} = \text{constant} \tag{9}$$

and

$$\beta = \beta_{\mathrm{a}}. \tag{10}$$

Therefore from (4), (7), (8) and (9) it follows that

$$m_{\mathrm{E}} = m_{\mathrm{a}}\,\beta_{\mathrm{a}}^{1-g} = \text{constant}\ \beta_{\mathrm{a}}^{1-g} = \text{constant}, \tag{11}$$

a condition implicit in the KP's $\mathscr{L}_{\mathrm{m}}$. At this point the only way *not* to have β_{a} = constant is to choose $g = 1$. But this is not what KP have. Since they chose $\pi(m) = -1$, i.e. that masses scale like inverse lengths, as they explicitly say in Section 2, it follows from (4) that they actually have $g = 2$, and so equation (11) implies

$$\beta_{\mathrm{a}} = \text{constant}. \tag{12}$$

Since in general $G_{\mathrm{E}} = G\beta^{g}$ with G_{E} = constant, it then follows that

$$G_{\mathrm{E}} = \beta_{\mathrm{a}}^{g}\,G_{\mathrm{a}} = \text{constant} = \text{constant}\ G_{\mathrm{a}}, \tag{13a}$$

i.e.

$$G_{\mathrm{a}} \sim \text{constant}. \tag{13b}$$

This concludes our proof. The matter Lagrangian chosen by KP plus $\pi(m) = -1$ inescapably lead to (13b) which is therefore *not* a new result as it is already contained in (6).

While it is clear that a scale invariant matter Lagrangian $\mathscr{L}_m$ must be added to the scale invariant gravitational Lagrangian $\mathscr{L}_g$ introduced by Canuto *et al.* (1977), it is clear that (6) *cannot be* the correct one and indeed it is not, as we have shown.

The misunderstanding of the true nature of scale invariance and the nature of (1) is further revealed by the insistence by KP that $\mathscr{L}_m$ 'must be given by (1)', that (1) is 'inescapable' and that only (1) will yield the correct geodesic equations. This last point is also incorrect as it will be shown in detail in a forthcoming paper by Canuto & Goldman (1981).

Finally, I would like to comment on equation (7) of KP's paper since it is internally inconsistent. KP write

$$G_E \sim \text{constant},$$

$$G_a \sim t^{-1} \tag{14a}$$

and

$$m_a \sim m_E \sim \text{constant}. \tag{14b}$$

We shall now show that (14a) and (14b) are mutually inconsistent. For (14b) to hold, we must have on the basis of the general relation (4) with $m = m_a$ and $\beta = \beta_a$,

$$\beta_a^{1-g} = \text{constant}. \tag{15}$$

Since KP have $g = 2$, it therefore follows that β_a = constant and so finally

$$G_a \sim \beta_a^{-g} \sim \text{constant} \tag{16}$$

which contradicts the second of (14a). Therefore (14a) and (14b) with $g = 2$ are mutually exclusive.

In conclusion, the paper by KP results from an uncritical borrowing of $\mathscr{L}_m$ from Dirac, a mixing up of different powers of G within the same context, and a confusion of how macroscopic and microscopic masses transform under scaling.

What the authors have done is to rediscover results *already* contained in their very starting Lagrangian and claim them as new.

References

Canuto, V. M., Adams, P., Hsieh, S.-H. & Tsiang, E., 1977. *Phys. Rev.*, **D16**, 1643.
Canuto, V. M. & Goldman, I., 1981. Preprint.
Kembhavi, A. K. & Pollock, M. D., 1981. *Mon. Not. R. astr. Soc.*, **197**, 1087.